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GRAVITY-INDUCED ANGULAR MOTION OF A SPINNING MISSILE

by

Charles H. Murphy

July 1971

U.S. ARMY ABERDEEN RESEARCH AND DEVELOPMENT CENTER
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Exterior Ballistics Laboratory

RDT&E Project No. 1T262301A201

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The usual analysis of the steady state angular motion of a dynamically stable spinning missile assumes a quasi-steady state calculation of a gravity-induced trim angle. A condition for the validity of this quasi-steady state assumption is derived. When this condition is not satisfied, the gravity-induced angular motion must be described differently for three distinct portions of the trajectory: the upleg, near apogee, and the downleg. The accuracy of this description is checked by comparison with numerical integrations. Finally the influence of cubic static and Magnus moments on the motion is determined and a revised point mass trajectory model is constructed.

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LIST OF SYMBOLS

C_D	drag coefficient
C_{L_α}	lift coefficient
C_{M_α}	static moment coefficient
$C_{M_{\dot\alpha}}$, C_{M_q}	damping moment coefficients
$C_{M_{p\alpha}}$	Magnus moment coefficient
G	$P g_{NT}\, \ell V^{-2}$
g_{NT}	component of gravity normal to the trajectory
g	gravity
H	$\rho \frac{S\ell}{2m} [C_{L_\alpha} - C_D - k_t^{-2} (C_{M_q} + C_{M_{\dot\alpha}})] - g\ell V^{-2} \sin \theta_T$
I_x, I_y	axial, transverse moments of inertia
K_j	amplitude of j-mode (j = 1, 2)
K_G	defined in Equation (3.6)
K_{2G}	defined in Equations (3.12, 4.3)
k_a	$(I_x/m\ell^2)^{\frac{1}{2}}$, axial radius of gyration
k_t	$(I_y/m\ell^2)^{\frac{1}{2}}$, transverse radius of gyration
ℓ	reference length
M	$\frac{\rho S\ell^3}{2I_y} C_{M_\alpha}$
M_o, M_2	cubic static moment coefficients in Equation (5.1)
m	mass
P	$\frac{I_x}{I_y} \frac{p\ell}{V}$, gyroscopic spin
p	roll rate
S	reference area

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LIST OF SYMBOLS (CONTINUED)

s	dimensionless distance along flight path, $\int_{t_o}^{t} (V/\ell)dt$
s_g	$P^2/4M$, gyroscopic stability factor
T	$\rho \frac{S\ell}{2m} [C_{L_\alpha} + k_a^{-2} C_{M_{p\alpha}}]$
T_o, T_2	cubic Magnus moment coefficients in Equation (5.1)
V	magnitude of velocity
$\hat{v}$, $\hat{w}$	$\hat{y}$, $\hat{z}$ components of the velocity vector
x_e, y_e, z_e	earth-fixed Cartesian coordinates
$\hat{x}$, $\hat{y}$, $\hat{z}$	fixed plane Cartesian coordinates
$\hat{\alpha}$, $\hat{\beta}$	angles of attack and sideslip
δ_g	G/M
δ	$\lvert\hat{\xi}\rvert$, sine of total angle of attack
θ_T	angle between trajectory and its projection on the horizontal plane
λ_j	damping rate of the j-mode amplitude (j = 1, 2), K_j'/K_j
$\hat{\xi}$	$\frac{\hat{v} + i\hat{w}}{V}$
$\hat{\xi}_g$	-G/M, steady state gravity-induced trim angle
$\hat{\xi}_G$	defined in Equation (3.3)
ρ	air density
ϕ_j	j-mode phase angle (j = 1, 2), $\phi_{jo} + \phi_j's$

LIST OF SYMBOLS (CONTINUED)

Superscripts

$(\ )^*$	$\rho \frac{S\ell}{2m}(\ )$
$(\ )'$	primes denote derivatives with respect to dimensionless arclength, s
$\hat{}$	components in fixed plane coordinate system
$(\dot{\ })$	derivatives with respect to time

Subscripts

a	evaluated at apogee
U	upleg
D	downleg

1. INTRODUCTION

The linear angular motion of missiles can usually be written as a sum of responses to various forcing functions and a solution involving the initial conditions. For a dynamically stable missile the effect of initial conditions quickly decays and the angular motion is controlled by the forcing functions, i.e. moments which do not depend on the missile's angle of attack or sideslip or their derivatives. For a slowly spinning missile the most important such forcing function is a constant pitch or yaw moment fixed on the missile and caused by either an intentional control surface deflection or an unintentional configurational asymmetry. The response to such a moment can take on large values when the pitch rate is near the roll rate and as a result it has been studied by a number of authors.[1-3*]

For a symmetric missile with a high spin rate, the forcing function has a magnitude which is proportional to the product of the spin-to-velocity ratio and the trajectory curvature, and has an axis of rotation which is perpendicular to the plane of the trajectory. For a constant amplitude moment and a linear static moment the response is a constant angle of sideslip. This trim sideslip angle causes the nose of a spinning shell to always point to the right and, thereby, produces a right deflection of the trajectory which is called drift.[4-5]

Since both the spin-to-velocity ratio and the trajectory curvature increases to a maximum at apogee, a maximum gravity-induced trim angle is predicted at apogee. This prediction assumes that a quasi-steady state calculation is appropriate and that the aerodynamic moments are linear. If either of these conditions is not satisfied, a complete six-degree-of-freedom numerical integration is usually required. It is the purpose of this paper to present a new simple approximation for

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**References are listed on page 37.*

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this gravity-induced angular motion which is valid for rapidly changing conditions near apogee. The effect of a nonlinear moment is incorporated by use of the quasilinear assumption which has been quite successful for the analysis of the transient motion.[6-7] Finally this approximation is used to obtain a revised version of a modified point mass trajectory.

II. EQUATIONS OF MOTION

We will make use of two Cartesian axis systems. The first is an earth-fixed system with the x_e-axis taken as the intersection of the horizontal plane with the plane of the trajectory, the z_e-axis aligned along the gravity vector and the y_e-axis specified by the right-hand rule. The second axis system has the $\hat{x}$-axis along the missile-axis of symmetry, the $\hat{z}$-axis in the plane of the trajectory pointing downward and the $\hat{y}$-axis determined by the right-hand rule. For this fixed plane axis system we make use of the complex angle of attack, $\hat{\xi}$, which is defined by the equation

$$\hat{\xi} \equiv \frac{\hat{v} + i\hat{w}}{V} = \sin\hat{\beta} + i\cos\hat{\beta}\sin\hat{\alpha} \qquad (2.1)$$

where $\hat{v}$, $\hat{w}$ are $\hat{y}$ and $\hat{z}$ components of the velocity vector and $\hat{\alpha}$, $\hat{\beta}$ are the angles of attack and sideslip. The magnitude of $\hat{\xi}$ is the sine of the angle between the missile's axis and the velocity vector and its orientation determines the orientation of the plane of this angle with respect to the horizontal. For a linear aerodynamics and small geometric angles $\hat{\xi}$ must satisfy the equation[7]

$$\hat{\xi}'' + (H - iP)\hat{\xi}' - (M \quad iPT)\hat{\xi} = G \qquad (2.2)$$

where the coefficients are defined in the List of Symbols.

The plane trajectory of a particle acted on by gravity and drag can be described by the equations

$$m\ddot{x}_e = -\frac{1}{2}\rho V^2 S C_D \left(\frac{\dot{x}_e}{V}\right) \tag{2.3}$$

$$m\ddot{z}_e = mg - \frac{1}{2}\rho V^2 S C_D \left(\frac{\dot{z}_e}{V}\right) \tag{2.4}$$

Introducing θ_T, the inclination of the trajectory with respect to the horizontal, these equations can be written in the form

$$\frac{V'}{V} = -C_D^* - g\ell V^{-2} \sin\theta_T \tag{2.5}$$

$$\theta_T' = -g\ell V^{-2} \cos\theta_T \tag{2.6}$$

where

$$C_D^* = \frac{\rho S \ell}{2m} C_D$$

Equations (2.5) - (2.6) can be integrated for constant C_D^* to give the velocity as a function of trajectory angle.

$$V = V_a \sec\theta_T \left\{1 - \frac{C_D^* V_a^2}{g\ell}\left[\tan\theta_T \sec\theta_T + \ln\tan\left(\frac{\theta_T}{2} + \frac{\pi}{4}\right)\right]\right\}^{-\frac{1}{2}} \tag{2.7}$$

where V_a is velocity at apogee.

The gravity terms in Equation (2.2) can now be approximated by θ_T if we assume a small angle of attack

$$g_{NT} = g \cos \theta_T \tag{2.8}$$

$$\therefore G = - P\theta_T' \tag{2.9}$$

The solution to Equation (2.2) for slowly varying coefficients is[6]

$$\hat{\xi} = K_1 e^{i\phi_1} + K_2 e^{i\phi_2} + \hat{\xi}_g \tag{2.10}$$

where

$$\phi_j' = \frac{1}{2}\left[P \pm [P^2 - 4M]^{\frac{1}{2}}\right] \tag{2.11}$$

$$\frac{K_j'}{K_j} = \lambda_j = - \frac{\left(H\phi_j' - PT + \phi_j''\right)}{2\phi_j' - P} \tag{2.12}$$

$$\hat{\xi}_g \equiv \frac{- G}{M + iPT} \doteq - \frac{G}{M} \tag{2.13}$$

The expression for the gravity-induced trim angle, $\hat{\xi}_g$, is based on the quasi-steady state assumption that G and M vary slowly during a cycle of the transient epicyclic motion given by the first two terms of Equation (2.10). (See Figure 1.)

III. GRAVITY-INDUCED TRIM WITHOUT DAMPING

Near apogee G varies rapidly as can be seen from its derivative for constant spin

$$G' = \left[-\frac{V'}{V} + \frac{\theta_T''}{\theta_T'}\right] G$$

$$= [3C_D^* + 4g\ell V^{-2} \sin\theta_T]G \qquad (3.1)$$

$$= [3C_D^* + 4g\ell V^{-2} \sin\theta_T]Pg\ell V^{-2} \cos\theta_T$$

The relative variation of G and G' are indicated in Figure 2. The maximum value of G occurs after apogee due to the action of drag. In order to obtain the angular response to G we make use of the method of variation of parameters for the simple case of constant λ_j's and ϕ_j''s and integrate the result by parts.

$$\hat{\xi} = K_1 e^{i\phi_1} + K_2 e^{i\phi_2} + \hat{\xi}_G \qquad (3.2)$$

$$\hat{\xi}_G = \hat{\xi}_g - \int_0^s \frac{\left[(\lambda_2 + i\phi_2')e^{(\lambda_1 + i\phi_1')(s-\hat{s})} - (\lambda_1 + i\phi_1')e^{(\lambda_2 + i\phi_2')(s-\hat{s})}\right]G'(\hat{s})d\hat{s}}{(M + iPT)[\lambda_1 - \lambda_2 + i(\phi_1' - \phi_2')]}$$

$$(3.3)$$

For a gyroscopically stabilized missile the fast rate ϕ_1' is usually much greater than the slow rate ϕ_2'. This is especially true near apogee where G' is large. Integrals of complex exponentials are inversely proportional to their frequencies and, thus, we can easily neglect the first term in the integral in comparison with the second term and for simplicity we will neglect λ_j in comparison with ϕ_j' in the multiplying coefficients but not in the exponential coefficients.

$$\hat{\xi}_G = -\delta_g + \frac{1}{M}\int_0^s \left[e^{(\lambda_2 + i\phi_2')(s - \hat{s})} \right] G'(\hat{s})d\hat{s} \qquad (3.4)$$

where $\delta_g = \frac{G}{M}$.

Although Equation (3.4) is a simple relation for the gravity-induced trim angle it is gravely limited by the restriction to a constant frequency. For most projectiles the apogee value of the gyroscopic stability factor usually exceeds ten when G' is large enough to affect Equation (3.4) and a quite simple expression for ϕ_2' can be written from Equation (2.11).

$$\phi_2' = \frac{P}{2}\left[1 - \left[1 - \frac{1}{s_g}\right]^{\frac{1}{2}}\right]$$

$$= \frac{M}{P}\left[1 + \frac{1}{4s_g} + \ldots\right] \doteq \frac{M}{P} \qquad (3.5)$$

where $s_g = P^2/4M$.

Since P is proportional to the spin-to-velocity ratio and the spin normally decays quite slowly due to viscous damping, P can grow quite rapidly on the upleg and, therefore, the assumption of constant ϕ_2' is not satisfied. Somewhat lengthy algebra shows that a good first approximation for the effect of varying frequency on the derivation of Equation (3.4) is to replace $\phi_2' s$ by $\phi_2 = \int_{s_a}^{s} \phi_2' \, ds$.

$$\therefore \hat{\xi}_G = -\delta_g + \delta_{ga} K_G e^{\lambda_2(s - s_a) + i\phi_2(s)} \qquad (3.6)$$

where

$$K_G = G_a^{-1} \int_0^s e^{-[\lambda_2(\hat{s} - s_a) + i\phi_2(\hat{s})]} G'(\hat{s})d\hat{s}$$

Equation (3.6) clearly reduces to the quasi-steady state relation when G' can be neglected. Indeed K_G can be neglected when the width of the humps of G' is large with respect to the wavelength of ϕ_2 .

The basic properties of Equation (3.6) can be determined if we consider the very simple case of zero drag, constant spin rate ($\dot{p} = 0$) and no aerodynamic damping. For zero drag Equation (2.7) reduces to

$$V = V_a \sec \theta_T \tag{3.7}$$

and

$$G' = 4(g\ell V_a^{-2})G_a \cos^6\theta_T \sin \theta_T \tag{3.8}$$

Equation (3.5), then, gives an approximation for ϕ_2'

$$\phi_2' = \frac{M}{P_a} \sec \theta_T = (g\ell V_a^{-2})\delta_{ga}^{-1} \sec \theta_T \tag{3.9}$$

The integral for K_G now assumes a very simple form for no damping

$$K_G(\phi_2, \delta_{ga}) = \delta_{ga} \int_{\phi_{20}}^{\phi_2} e^{-i\hat{\phi}_2} f(\theta_T)d\hat{\phi}_2 \tag{3.10}$$

where $f(\theta)_T = 4 \cos^7 \theta_T \sin \theta_T$.

Finally a relationship between θ_T and ϕ_2 can be obtained from Equations (2.6) and (3.9)

$$\phi_2 = -\delta_{ga}^{-1} \tan\theta_T[3 + \tan^2\theta_T]/3 \qquad (3.11)$$

$f(\theta_T)$ is plotted versus ϕ_2 for various values of δ_{ga} in Figure 3.

A brief examination of K_G shows that it is essentially constant for ϕ_2 outside the interval $(-2\pi, 2\pi)$. On the upleg portion of the trajectory ($\phi_2 < -2\pi$) K_G is zero while on the downleg portion ($\phi_2 > 2\pi$) it has a zero real part. This situation can be summarized by the following equation

$$\hat{\xi}_G = -\delta_g + \delta_{ga} K_G(\delta_{ga}, \phi_2)e^{i\phi_2} \qquad (3.12)$$

where

$$K_G = 0 \qquad \phi_2 < -2\pi \qquad \text{upleg}$$

$$K_G = \delta_{ga} \int_{-2\pi}^{\phi_2} e^{-i\hat{\phi}_2} f(\theta_T)d\hat{\phi}_2$$

$$-2\pi < \phi_2 < 2\pi \qquad \text{near apogee}$$

$$K_G = K_G(\delta_{ga}, 2\pi) \qquad 2\pi < \phi_2 \qquad \text{downleg}$$

$$= iK_{2G}(\delta_{ga})$$

where

$$K_{2G} = -\delta_{ga} \int_{-2\pi}^{2\pi} f(\theta_T) \sin\hat{\phi}_2 \, d\hat{\phi}_2$$

$K_{2G}(\delta_{ga})$ is given as a function of δ_{ga} in Figure 4. An important feature of this curve is that K_{2G} is quite small for δ_{ga} less than .15 and thus we would expect the quasi-steady state results to be good when the predicted apogee steady state angle is less than 8°. When the steady state prediction exceeds this value, Equation (3.12) or the more accurate Equation (3.6) should be used. K_{2G} can be identified as a fraction of δ_{ga} which appears impulsively at the apogee in the slow mode when the gravity forcing function is rapidly varying in a period of the slow mode.

In Figure 5 the combined pitching and yawing motion for a missile with a large maximum gravity-induced trim (β = 20°) is shown. The parameters used in this exact integration are given in Table I. From Figure 4 we see that K_{2G} is .76 and, therefore, the amplitude of the slow mode component of Equation (3.12) is (.76)(20°) or 15°.

This motion starts out as an epicyclic motion induced by an initial angular velocity. During the first three seconds the center of this epicycle moves to the right; then it moves up as well as continuing its right motion until eight seconds. There then appears a rough reversal of this process until eleven seconds is reached. After this point a new epicyclic motion is established with a much larger slow mode motion with an amplitude of 12°. This qualitative behavior is precisely that predicted by Equation (3.12) with the near apogee motion occurring between three and eleven seconds. The terminal slow mode amplitude of 12° is quite consistent with an apogee value of 15° when the influence of aerodynamic damping is computed.

IV. GRAVITY-INDUCED TRIM WITH DAMPING

The effect of constant damping is included in Equation (3.6). Near apogee ϕ_2' is much smaller than P and from Equation (2.12) we see that a good approximation for λ_2 is -T which can be constant for near apogee flight For this case Equation (3.12) takes on the revised form

$$\hat{\xi}_G = -\delta_g + \delta_{ga} K_G(\delta_{ga}, \phi_2) e^{-T(s - s_a) + i\phi_2} \tag{4.1}$$

where

$$K_G = \delta_{ga} \int_{-2\pi}^{\phi_2} e^{T(\hat{s} - s_a) - i\hat{\phi}_2} f(\theta_T) d\hat{\phi}_2 \qquad -2\pi < \phi_2 < 2\pi$$

On the downleg portion ($s > s_D$) of the flight ϕ_2' grows and H and T vary as the Mach number increases. During this portion of flight G' is quite small and the integral in Equation (3.6) becomes constant. We, therefore, assume the major effect of G' is to specify an initial value of K_2 and use Equation (2.12) to predict the influence of varying λ_2 .

$$\hat{\xi}_G = -\delta_g + K_2 e^{i(\phi_2 + \phi_{2G})} \tag{4.2}$$

$$K_2 = \delta_{ga} K_{2G} \exp\left\{ \int_{s_a}^{s} \lambda_2 \, d\hat{s} \right\} \tag{4.3}$$

where

$$K_{2G} e^{i\phi_{2G}} = G_a^{-1} \int_{s_U}^{s_D} e^{T(\hat{s} - s_a) - i\phi_2(\hat{s})} \, G' \, d\hat{s}$$

For zero drag a simple expression for K_{2G} can be obtained.

$$K_{2G} = \left| \delta_{ga} \int_{-2\pi}^{2\pi} e^{T(\hat{s} - s_a) - i\hat{\phi}_2} f(\theta_T) \, d\hat{\phi}_2 \right| \tag{4.4}$$

If $|T/\phi_2'| < 0.1$, actual numerical calculations show that K_{2G} is within .02 of its value for $T = 0$ and, hence, Figure 4 can be used to obtain K_{2G} as a function of δ_{ga} .

V. NONLINEAR ANALYSIS

The usual quasi-linear analysis[6-9] has been applied primarily to the angular motions of symmetric missiles with no moment forcing functions. This analysis has recently been extended to include the forcing function associated with slight configurational asymmetries.[3] The latter treatment can be easily extended to include gravity-induced angular motion away from apogee.[6,10] In this section we will outline the appropriate analysis and give the results for cubic static and Magnus moments.

For this case Equation (2.2) becomes

$$\hat{\xi}'' + (H - iP)\,\hat{\xi}' - [M_o + M_2\,\delta^2 + iP\,(T_o + T_2\,\delta^2)]\,\hat{\xi} = G \qquad (5.1)$$

where

$$\delta^2 = |\hat{\xi}|^2$$

A solution of the form of Equation (4.2) is assumed and substituted in Equation (5.1). The resulting equation is divided by $K_2 e^{i\phi_2}$ and averaged over a distance which is large with respect to the wavelength of the slow rate to yield quasi-linear values of λ_2 and ϕ_2'.

$$\lambda_2 = -\frac{H\phi_2' - P\,[T_o + T_2\delta_{e2}^2] + \phi_2''}{2\phi_2' - P} \qquad (5.2)$$

$$\phi_2' = 1/2\left[P - \sqrt{P^2 - 4\,[M_o + M_2\delta_{e2}^2]}\right] \qquad (5.3)$$

where

$$\delta_{e2}^2 = K_2^2 + 2\delta_g^2 + 2K_1^2$$

If the resulting equation is divided by $K_1 e^{i\phi_1}$, similar relations for the high frequency motion follow. Finally the equation can be averaged as it is to yield a quasi-linear relation for the gravity-induced trim.

$$-\left[M_o + M_2\delta_{e3}^2 + iP\left(T_o + T_2\delta_{e3}^2\right)\right]\hat{\xi}_g = G \tag{5.4}$$

where

$$\delta_{e3}^2 = \delta_g^2 + 2K_2^2 + 2K_1^2$$

Since the imaginary part of the coefficient of $\hat{\xi}_g$ is usually less than a quarter of the real part, it affects the orientation of ξ_g much more than it affects its magnitude, δ_g. A simple equation for δ_g can be written.

$$\delta_g = \frac{G}{M_o + M_2(\delta_g^2 + 2K_2^2)} \tag{5.5}$$

On the downleg Equations (5.2 - 5.3) can be used in Equation (4.3) to calculate the magnitude of the slow mode motion which has been initiated by G' at the apogee. The orientation of the slow mode motion can be obtained by integrating Equation (5.3).

The nonlinear analysis for near apogee motion is much more difficult since δ_g varies rapidly during a cycle of ϕ_2. An estimate of the effect of a cubic static moment can be made for small values of K_2. The steady state formulas for δ_g then reduce to a cubic equation.

$$\delta_g = \frac{G}{M_o + M_2 \delta_g^2} \qquad (5.6)$$

The slow frequency, which assumes the form

$$\phi_2' = \left[\frac{M_o + 2M_2 \delta_g^2}{P_a}\right] \sec \theta_T , \qquad (5.7)$$

varies in response to the nonlinearity as δ_g grows from zero to δ_{ga}. If the nonlinearity term in Equation (5.7) is replaced by its average, this equation can be reduced to Equation (3.9) of the linear theory.

$$\phi_2' = \left[\frac{M_o + M_2 \delta_{ga}^2}{P_a}\right] \sec \theta_T = (g\ell V_a^{-2})\ \delta_{ga} \sec \theta_T \qquad (5.8)$$

Thus, an approximation for K_G and K_{2G} when the static moment is a cubic function can be made by using Equation (3.12) with δ_{ga} given by Equation (5.6) evaluated at the apogee.

VI. A REVISED POINT MASS TRAJECTORY

For many years ordnance firing tables were computed by use of the point mass Equations (2.3 - 2.4). These equations completely neglect induced drag due to $\hat{\xi}_G$ as well as lateral drift caused by this angle. The induced drag is accounted for by adjusting C_D by a form factor which is a function of θ_T and is determined by full-range firing. Drift is measured by full-range firings and numerically interpolated for firing table use.

Recently a modified point mass analysis has been developed[11] which includes the effects of the steady state gravity-induced trim $\hat{\xi}_g = -\delta_g$. This trim angle modified the drag coefficient and induced a lateral deflection.

$$C_D = C_{D_o} + C_{D_{\delta^2}} \delta^2_g \qquad (6.1)$$

$$m\ddot{y}_e = \frac{1}{2} \rho V^2 S C_{L_\alpha} \delta_g \qquad (6.2)$$

This modified point mass trajectory has the advantage of retaining the major trajectory contribution of the angular motion without requiring the use of the very small integration interval associated with an exact integration of Equation (2.2). It is valid for a dynamically stable missile and slowly varying G.

The theory of this report can be used to construct an improved version of the modified point mass trajectory which could be called a revised point mass trajectory. Since the motion near and after apogee involves the slow frequency, an integration interval small with respect to the slow mode's period is needed. The integration interval required for Equation (2.2) is small with respect to the fast mode's period and is, therefore, much smaller than that required for the revised point mass trajectory. Thus the revised point mass trajectory requires much less computer time than the exact six-degree-of-freedom trajectory.

It has been shown[7] that only the average of δ^2 need be considered for the drag force. If we can neglect the effect of K_G on drag near apogee where the total drag force is small, only the drag on the down-leg need be revised.

$$C_D = C_{D_o} + C_{D_{\delta^2}} (\delta_g^2 + K_2^2) \tag{6.3}$$

The limitation imposed on the integration interval by the slow mode motion can now be eliminated if the lateral deflection due to K_G can be neglected. The lateral deflection due to K_G is caused by an angle which is constantly changing direction. The average value of this deflection angle can be estimated by calculating the jump angle[12] for a projectile performing coning motion of magnitude K_2 and frequency ϕ'_{2a}.

$$\text{Jump angle} = \frac{C^*_{L_\alpha} K_2}{\phi'_{2a}} \tag{6.4}$$

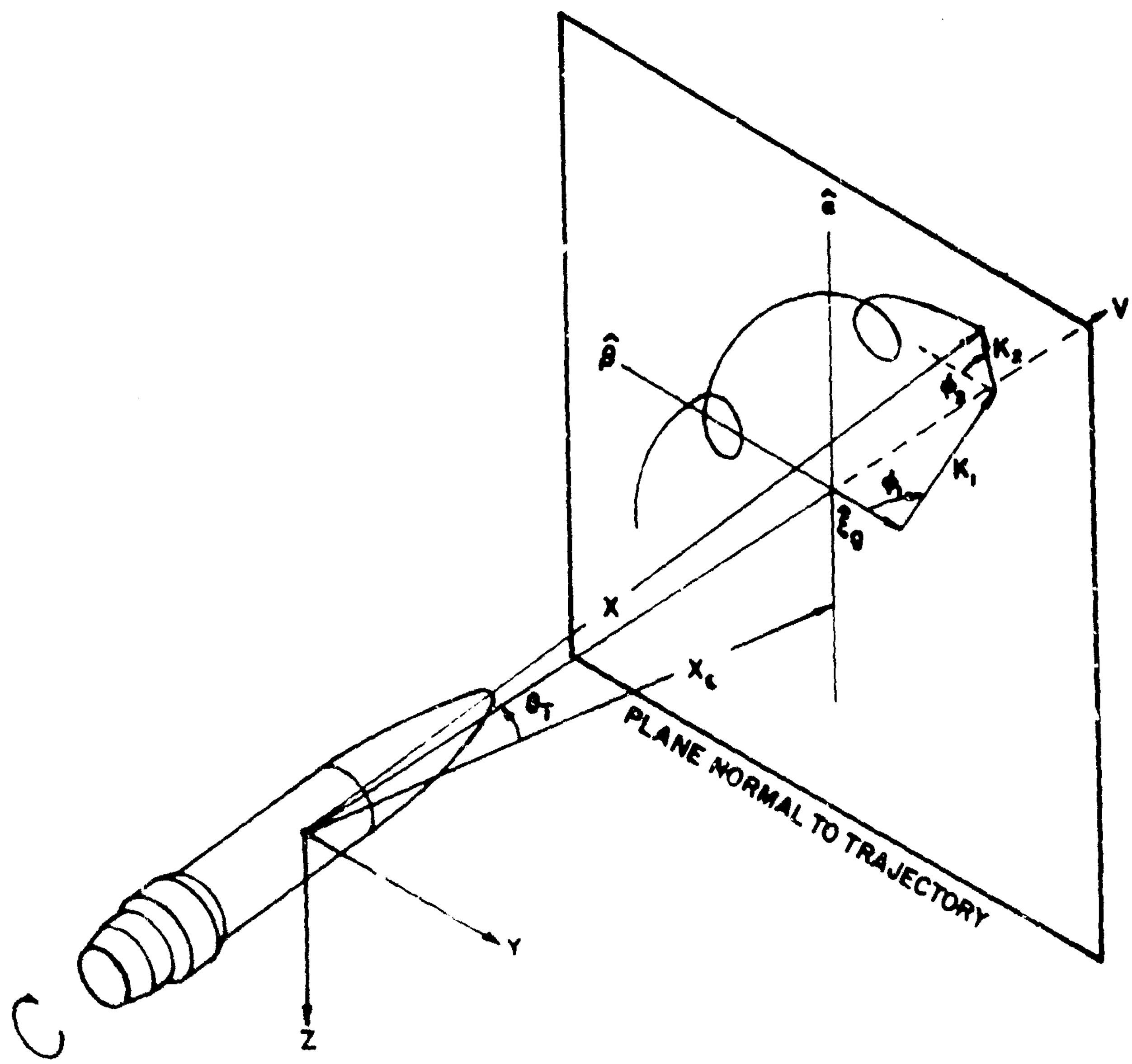
This jump angle is a right deflection angle of the impact point with respect to the apogee. The deflection angle with respect to the gun is one-half this angle and usually quite small. The refined point mass trajectory, then, requires the same integration interval as the modified point mass trajectory. If the effect of K_G on drag near apogee is required, a smaller integration interval will be required for this small portion of the trajectory.

VII. COMPARISON WITH EXACT THEORY EVALUATIONS

In order to make a direct comparison with exact calculations initial conditions of $\xi_o = -\delta_{g_o}$, $\xi'_o = -\delta'_{g_o}$ were used with the other parameters of Table I to give an angular motion without a transient epicycle. The total angle of attack variation with time for these conditions is shown in Figure 6 and is compared with the quasi-steady state δ_g and the angular motion given by Equation (3.12). We see that the prediction of Equation (3.12) is much better than that of the quasi-steady state theory but it does overestimate α_t by 35%.

The calculations based on Equation (3.12) can be considerably simplified if the near apogee motion is approximated by a discontinuous jump at apogee from the - δ_g motion before apogee to the - δ_g + K_2 exp $(i\phi_2)$ motion after apogee. This calculation which considers only two sections of the trajectory is also given in Figure 6 and with the exception of a region very close to apogee it is seen to be a good approximation to the three-section theory.

Finally the effect of damping is calculated through Equations (4.1 - 4.3). The two- and three-section calculations were repeated for nonzero damping and are plotted in Figure 6. Here we see that the theory underestimates α_t by about 15% near t = 13 sec. This discrepancy, however, is entirely due to calculating K_{2G} over a two-cycle interval, i.e. one cycle on both sides of apogee. K_{2G} was then calculated over a four-cycle interval (two cycles on both sides of apogee) and the result is plotted as Figure 4. This shows a difference of about 5%. The two-section calculation is repeated in Figure 6 using the four-cycle integration value of K_{2G} and we see the agreement with the exact curve to be quite good.



$$\hat{\xi} \doteq \beta + i\hat{\alpha} = \xi_g + K_1 e^{i\phi_1} + K_2 e^{i\phi_2}$$

Figure 1. Angular Motion of Spinning Projectile

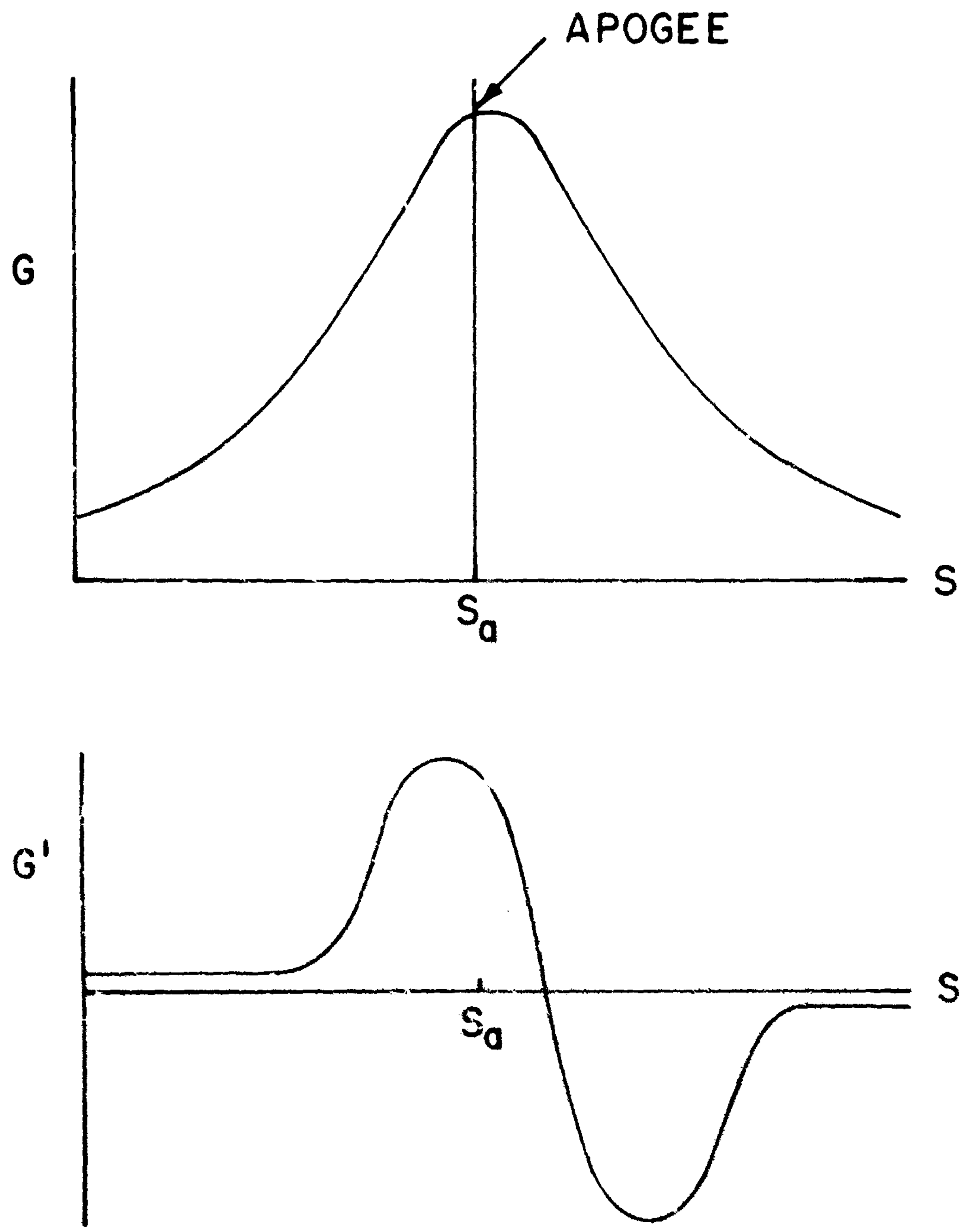


Figure 2. Variation of G and G' over the F.ight Path

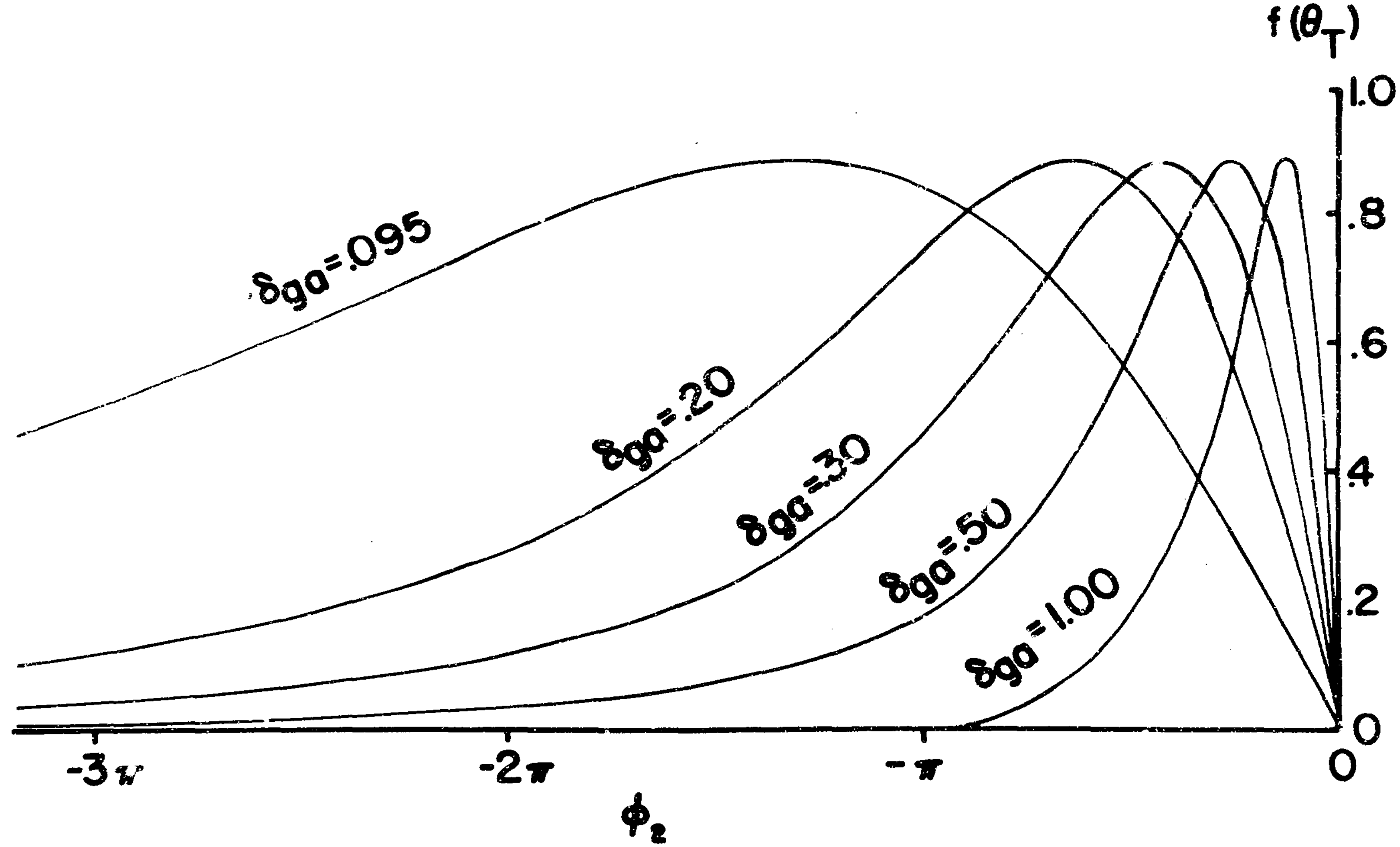


Figure 3. Variation of $f(\theta_T)$ as a Function of the Slow Mode Phase Shift from Apogee for Several Values of δ_{ga}

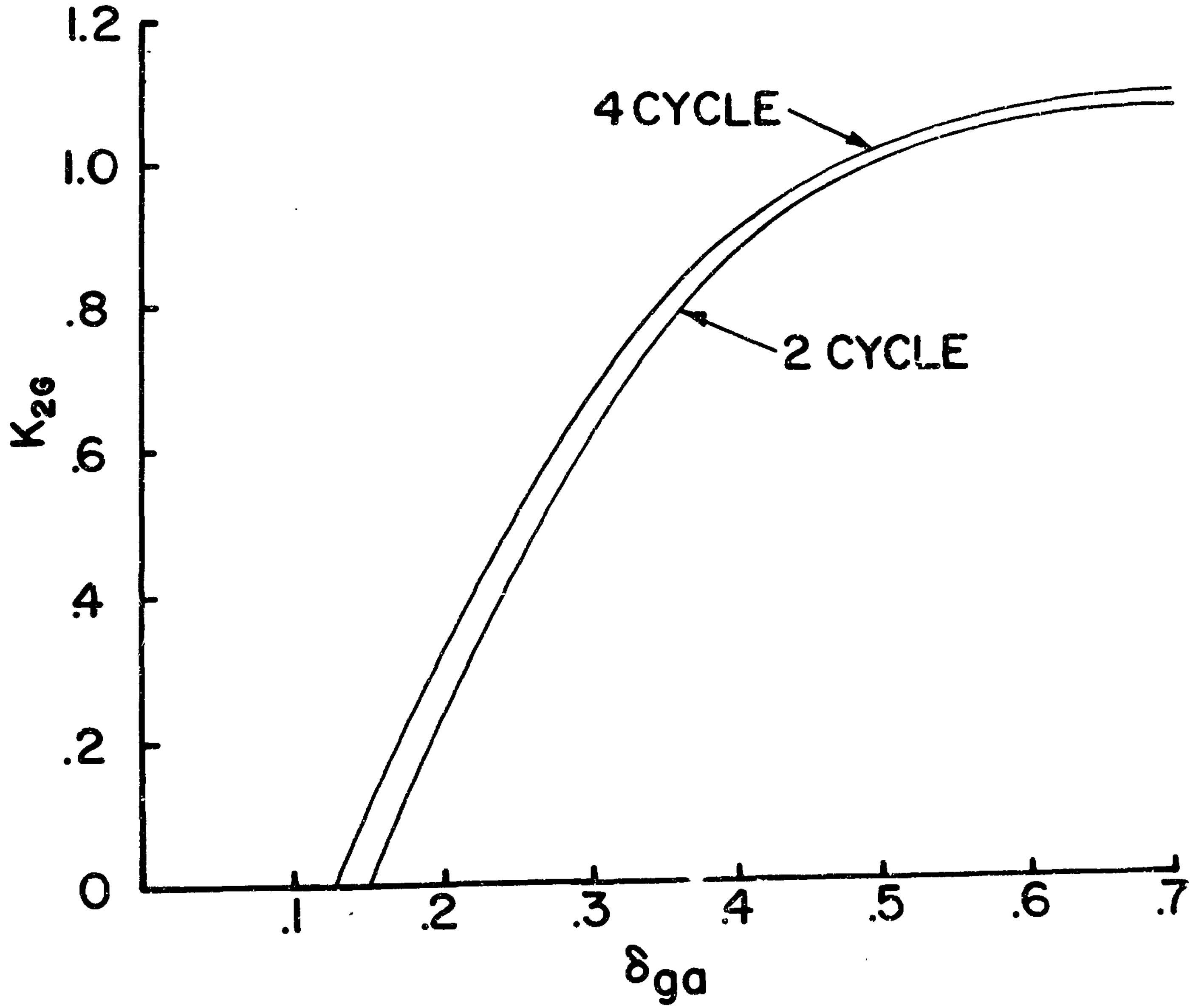


Figure 4. K_{2G} versus δ_{ga} for $\lambda_2 = 0$. (For $|T_2/\phi_2'| < .1$ the change in K_{2G} is less than .01)

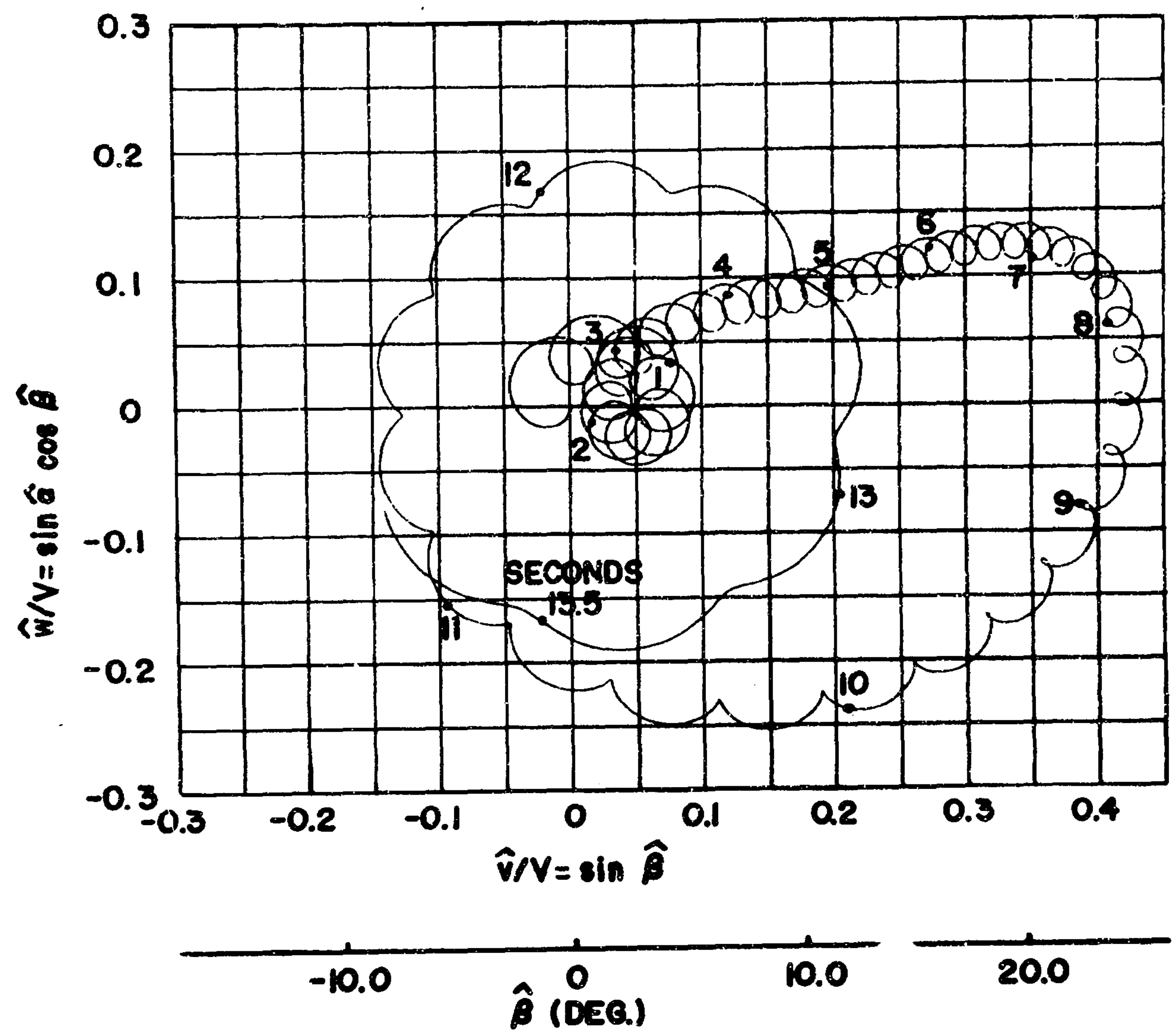


Figure 5. Pitching and Yawing Motion of 4.2 Mortar Shell

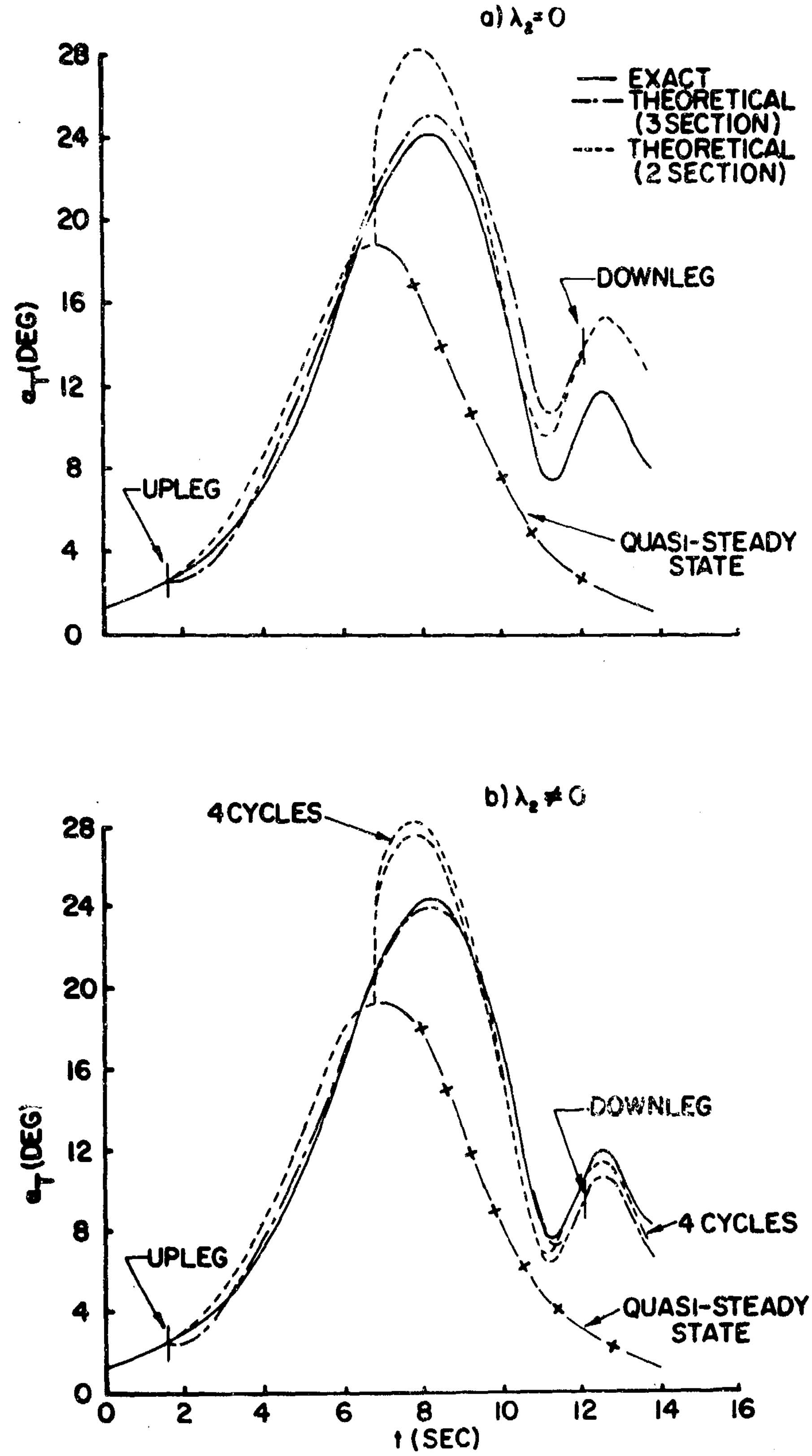


Figure 6. Comparison of Exact Values of Total Angle of Attack with Various Theoretical Approximations

Table I. Parameters for Exact Integration

V_o	$= 255$ ft/sec	δ_{ga}	$= .335$ ($\hat{\beta} = 20°$)
$\theta_T(0)$	$= 60°$	δ_{go}	$= .021$ ($\hat{\beta}_g = 1.2°$)
M	$= 1.5 \times 10^{-4}$	ϕ'_{1a}	$= .071$
T	$= 1.5 \times 10^{-4}$	ϕ'_{2a}	$= .002$
H	$= 2.9 \times 10^{-4}$	$\hat{\xi}'(0)$	$= -i\,(.5)$ rad/sec
P_o	$= .036$	$\xi(0)$	$= 0$
C_D^*	$= 0$		

REFERENCES

1. J. D. Nicolaides, "On the Free Flight Motion of Missiles Having Slight Configurational Asymmetries," Ballistic Research Laboratories Report No. 858, June 1953, AD 26405; also IAS Preprint 395, January 1953.

2. D. A. Price, Jr., "Sources, Mechanisms and Control of Roll Resonance Phenomena for Sounding Rockets," Proceedings of AIAA Sounding Rocket Vehicle Technology Specialist Conference, March 1967, pp. 224-235.

3. C. H. Murphy, "Nonlinear Motion of a Missile with Slight Configurational Asymmetries," *Journal of Spacecraft and Rockets*, Vol. 8, No. 3, March 1971, pp. 259-263; also published as Ballistic Research Laboratories Memorandum Report No. 2036, May 1970, AD 870704.

4. R. H. Fowler, E. G. Gallop, C. N. H. Lock and H. W. Richmond, "The Aerodynamics of a Spinning Shell," *Phil. Trans. Roy. Soc. London A*, Vol. 221, 1920, pp. 295-387.

5. E. J. McShane, J. L. Kelley and F. V. Reno, *Exterior Ballistics*, University of Denver Press, Denver, 1953.

6. C. H. Murphy, "Predictions of the Motion of Missiles Acted on by Nonlinear Forces and Moments," Ballistic Research Laboratories Report No. 995, October 1956, AD 122221.

7. C. H. Murphy, "Free Flight Motion of Symmetric Missiles," Ballistic Research Laboratories Report No. 1216, July 1963, AD 442757.

8. C. H. Murphy, "Quasi-Linear Analysis of the Nonlinear Motion of a Nonspinning Symmetric Missile," *Journal of Applied Mathematics and Physics (ZAMP)*, Vol. 14, September 1963, pp. 630-643.

9. C. H. Murphy, "Angular Motion of a Re-Entering Symmetric Missile," *AIAA Journal*, Vol. 3, July 1965, pp. 1275-1282.

10. W. R. Haseltine, "Existence and Stability Theorems for Exterior Ballistics," *SIAM Journal Control*, Vol. 6, 1968, pp. 386-400.

11. R. F. Lieske and M. L. Reiter, "Equations of Motion for a Modified Point Mass Trajectory," Ballistic Research Laboratories Report No. 1314, March 1966, AD 485869.

12. C. H. Murphy and J. W. Bradley, "Jump Due to Aerodynamic Asymmetry of a Missile with Varying Roll Rate," Ballistic Research Laboratories Report No. 1077, May 1959, AD 219312.

APPENDIX A

DERIVATION OF EQUATION (2.2)

Most of the relations given in Reference 7 are for near horizontal trajectories: $\theta_T = 0$. These relations neglect the trajectory component of gravity and use an approximation for G. The more exact equations are given in Reference 5 but in a much different notation. It is the purpose of this note to rederive the equations of Reference 7 for non-horizontal trajectories. We will refer frequently to equations in Reference 7 and to avoid confusion will add the letter R and chapter number to equation numbers from this reference to distinguish them from equations in the body of this report.

Equation (R5.2.2) is the drag equation for a horizontal trajectory. For more general trajectories, the trajectory component of gravity must be included and, thus, this equation must be replaced by Equation (2.5)

$$\frac{V'}{V} = - C_D^* - g\ell V^{-2} \sin \theta_T \tag{A-1}$$

The roll equation (Equation (R5.6)) must now be modified using the more general drag equation:

$$\phi'' + K_p \phi' - K_\delta = 0 \tag{A-2}$$

$$\text{where} \quad K_p = - \left[k_a^{-2} C_{\ell_p}^* + C_D^* + g\ell V^{-2} \sin \theta_T \right]$$

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$$K_\delta = k_a^{-2}\, \delta_f\, C^*_{\ell_\delta}$$

$$C_\ell = \delta_f\, C_{\ell_\delta} + \frac{p\ell}{V}\, C_{\ell_p}$$

C_ℓ is the roll moment coefficient

For a body of revolution, $K_\delta = 0$ and

$$\phi' \equiv \frac{p\ell}{V} = \phi'_o\, e^{-\int_o^s K_p ds_1} \qquad \text{(A-3)}$$

Equations (R6.4.24-R6.4.25), which are the equations for transverse motion with arbitrary aerodynamic force and moment, can now be re-written using the more general drag equation:

$$\tilde{\xi}' - \left(C_D^* + g\ell V^{-2} \sin \theta_T\right) \tilde{\xi} - i\gamma\, \tilde{\mu}$$

$$= C^*_{\tilde{Y}} + i\, C^*_{\tilde{Z}} + \left(g_{\tilde{y}} + i\, g_{\tilde{z}}\right) \ell V^{-2} \qquad \text{(A-4)}$$

$$\tilde{\mu}' - \left(C_D^* + g\ell V^{-2} \sin \theta_T + i\,P\right) \tilde{\mu}$$

$$= \left(C_{\tilde{m}}^* + i\,C_{\tilde{n}}^*\right) k_t^{-2} \qquad \text{(A-5)}$$

where $\tilde{\xi} = \dfrac{\tilde{v} + i\tilde{w}}{V}$

$$\tilde{\mu} = \frac{(\tilde{q} + i\tilde{r})\ell}{V}$$

$$\gamma = \frac{u}{V}$$

$(u, \tilde{v}, \tilde{w})$ are components of the velocity vector in non-rolling coordinates

$(p, \tilde{q}, \tilde{r})$ are components of the angular velocity vector in non-rolling coordinates

$C_{\tilde{Y}}$, $C_{\tilde{Z}}$ are $\tilde{Y}$ and $\tilde{Z}$ components of the non-dimensional aerodynamic force

$C_{\tilde{m}}$, $C_{\tilde{n}}$ are $\tilde{Y}$ and $\tilde{Z}$ components of the non-dimensional aerodynamic moment; and

$g_{\tilde{y}}$, $g_{\tilde{z}}$ are $\tilde{Y}$ and $\tilde{Z}$ components of the gravitational acceleration vector

The non-rolling coordinate axes $X\ \tilde{Y}\ \tilde{Z}$ are defined as follows: 1) the X-axis is aligned along the projectile's axis of symmetry; 2) the $\tilde{Y}$ and $\tilde{Z}$ are the orthogonal axes of a right-handed Cartesian set which move so that the X-component of the angular velocity of the coordinate system is zero. Thus the axis system pitches and yaws with the projectile but has a zero roll rate.

A good approximation for the aerodynamic force is a linear normal force.

$$C_{\tilde{Y}} + i\, C_{\tilde{Z}} = -\, C_{N_\alpha}\, \tilde{\xi} \tag{A-6}$$

Equations (A-4) and (A-6) can now be combined to yield a revised version of Equation (R6.6.1)

$$\tilde{\xi}' - i\gamma\tilde{\mu} = -\left(C^*_{N_\alpha} - C^*_D\right)\tilde{\xi} + \left(g_{\tilde{y}} + i\, g_{\tilde{z}} + g \sin\theta_T\, \tilde{\xi}\right)\ell V^{-2} \tag{A-7}$$

The usual linear expression for the aerodynamic moment is

$$C_{\tilde{m}} + i\, C_{\tilde{n}} = \left[\left(\frac{p\ell}{V}\right) C_{M_{p\alpha}} - i\, C_{M_\alpha}\right]\tilde{\xi} + C_{M_q}\,\tilde{\mu} - i\, C_{M_{\dot{\alpha}}}\,\tilde{\xi}' \tag{A-8}$$

Combining Equations (A-5) and (A-8) we can obtain

$$\tilde{\mu}' - i P \tilde{\mu} = k_t^{-2}\left[\left(\frac{p\ell}{V}\right) C^*_{M_{p\alpha}} - i C^*_{M_\alpha}\right] \tilde{\xi}$$

$$+ \left(k_t^{-2} C^*_{M_q} + C^*_D + g\ell V^{-2} \sin \theta_T\right) \tilde{\mu}$$

$$- i k_t^{-2} C^*_{M_{\dot{\alpha}}} \tilde{\xi}' \qquad \text{(A-9)}$$

Equations (A-7) and (A-9) can now be used to eliminate $\tilde{\mu}$ and $\tilde{\mu}'$.

$$\tilde{\xi}'' + \left(H - \frac{\gamma'}{\gamma} - i P\right) \tilde{\xi}' - (M + i P T)\tilde{\xi} = \tilde{G} \qquad \text{(A-10)}$$

where

$$H = \gamma C^*_{L_\alpha} - C^*_D - k_t^{-2} \left(C^*_{m_q} + \gamma C^*_{M_{\dot{\alpha}}}\right) - g\ell V^{-2} \sin \theta_T$$

$$M \doteq \gamma k_t^{-2} C^*_{M_\alpha} - \gamma \left(C^*_{L_\alpha}\right)' \doteq \gamma k_t^{-2} C^*_{M_\alpha}$$

$$T = \gamma \left[C^*_{L_\alpha} + k_a^{-2} C^*_{M_{p\alpha}}\right]$$

$$\tilde{G} = \tilde{G}_1' - \left[k_t^{-2} C^*_{M_q} + C^*_D + g\ell V^{-2} \sin \theta_T + \frac{\gamma'}{\gamma} + i\,P\right]\tilde{G}_1$$

$$\tilde{G}_1 = \left(g_{\tilde{y}} + i\, g_{\tilde{z}} + g \sin \theta_T\, \tilde{\xi}\right) \ell V^{-2} \qquad \text{and}$$

$$\gamma C_{L_\alpha} = C_{N_\alpha} - C_D$$

For small geometric angles ($\gamma = 1$, $\gamma' = 0$) the definitions of H, M, and T reduce to those used in the text of this report. In order to differentiate $\tilde{G}_1$, we use the identity

$$\dot{\vec{g}} = \left(\dot{g}_x, \dot{g}_{\tilde{y}}, \dot{g}_{\tilde{z}}\right) + \left(0, \tilde{q}, \tilde{r}\right) \times \vec{g} \qquad \text{(A-11)}$$

Therefore,

$$g_{\tilde{y}}' + i\, g_{\tilde{z}}' = i\, g_x\, \tilde{\mu} \qquad \text{(A-12)}$$

Equations (A-7) and (A-12) and some algebraic effort yield a good linear approximation for $\tilde{G}$:

$$\tilde{G} = - \left[k_t^{-2} C^*_{M_q} - C^*_D - 2g\ell V^{-2} \sin \theta_T + i\,P\right] \tilde{G}_1$$

$$\doteq - \; i\,P\,\tilde{G}_1 \qquad \text{(A-13)}$$

For small amplitude motion the distinction between non-rolling coordinates and the fixed plane coordinates vanishes so that tilde superscripts on ξ, G_1, y and z can be replaced by carats and $\tilde{G}$ replaced by G.

$$G = P\left(g_{\hat{z}} - i\, g \sin \theta_T\, \hat{\xi}\right)\ell V^{-2} \qquad \text{(A-14)}$$

Finally, the gravitational acceleration perpendicular to the trajectory can be expressed as

$$g_{\hat{z}} = g \cos\left(\hat{\alpha} + \theta_T\right)$$

$$= g\left[\cos \theta_T \cos \hat{\alpha} - \sin \theta_T \sin \hat{\alpha}\right]$$

$$\doteq g_{NT} - \left(g \sin \theta_T\right)\hat{\alpha} \qquad \text{(A-15)}$$

where $g_{NT} = g \cos \theta_T$

$$\therefore G = P\left[g_{NT} - ig \sin \theta_T\, \hat{\beta}\right]\ell V^{-2}$$

$$\doteq P\, g_{NT}\, \ell V^{-2} \qquad \text{(A-16)}$$

APPENDIX B

DERIVATION OF EQUATION (3.6)

Equation (3.6) can be derived quite quickly by formally following the usual steps for the well-known method of variation of parameters (see, for example, pages 71-72 of Reference 7). The solution to the homogeneous form of Equation (2.2) with slowly varying coefficients is[7]

$$\hat{\xi} = A_1 \, e^{B_1} + A_2 \, e^{B_2} \qquad \text{(B-1)}$$

where $B_j = \int_0^s \lambda_j \, d\hat{s} + i \, \phi_j$

A_j are constants; and

λ_j and ϕ_j' satisfy Equations (2.11-2.12)

The complex parameters A_j are now made functions of s to construct a particular integral for the inhomogeneous term, G. Differentiating Equation (B-1), we have

$$\hat{\xi}' = A_1 \, B_1' \, e^{B_1} + A_2 \, B_2' \, e^{B_2} + A_1' \, e^{B_1} + A_2' \, e^{B_2} \qquad \text{(B-2)}$$

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Since the differential equation is only one condition on the two A_j functions, a second condition can be selected. A good choice is

$$A_1' \, e^{B_1} + A_2' \, e^{B_2} = 0 \qquad \text{(B-3)}$$

Equation (B-2) can now be differentiated to yield

$$\hat{\xi}'' = A_1\left[B_1'' + \left(B_1'\right)^2\right] e^{B_1} + A_2\left[B_2'' + \left(B_2'\right)^2\right] e^{B_2}$$

$$+ A_1' \, B_1' \, e^{B_1} + A_2' \, B_2' \, e^{B_2} \qquad \text{(B-4)}$$

Equations (B-1 - B-4) are now substituted in Equation (2.2) and coefficients of A_1, A_2, A_1', and A_2' determined. Since Equation (B-1) for constant A_j is a good approximate solution to the homogeneous part of Equation (2.2), the coefficients of A_1 and A_2 must vanish.

$$\therefore \; A_1' \, B_1' \, e^{B_1} + A_2' \, B_2' \, e^{B_2} = G \qquad \text{(B-5)}$$

Equations (B-3) and (B-5) can now be solved for A_1 and A_2 and the result substituted in Equation (B-1) to obtain a particular integral of Equation (2.2). Indeed, this was the method used to obtain Equation (3.3) for constant damping rates and frequencies.

Our primary interest is in the low frequency mode identified by the 2 subscript and so we solve Equations (B-3) and (B-5) for A_2 .

$$A_2 = - \int_0^s \left(G\right)\left(B_1' - B_2'\right)^{-1} e^{-B_2} d\hat{s}$$

$$= \left.\frac{G\, e^{-B_2}}{\left(B_1' - B_2'\right) B_2'}\right]_0^s - \int_0^s \left[G/\left(B_1' - B_2'\right)B_2'\right]' e^{-B_2} d\hat{s}$$

(B-6)

A similar expression can be obtained for A_1 . When these are substituted in Equation (B-1), the upper bound of the first term in Equation (B-6) gives a slowly varying contribution to the quasi-steady state ξ_g, and the lower bound appears as a constant multiplied by e^{B_2} . For dynamically stable missiles, the λ_j are negative and this lower bound term quickly damps out. The integral contribution of the fast motion is neglected as it was for constant frequencies and we have

$$\hat{\xi}_G = \hat{\xi}_g - e^{B_2} \int_0^s \left[G/\left(B_1' - B_2'\right) B_2'\right]' e^{-B_2(\hat{s})} d\hat{s} \qquad \text{(B-7)}$$

The integral is small everywhere except near the summit. Here the gyroscopic stability factor is large.

$$\therefore B_j \doteq i\,\phi_j' = \frac{P}{2}\left[1 \pm \left[1 - \frac{1}{s_g}\right]^{\frac{1}{2}}\right] \qquad \text{(B-8)}$$

$$\left(B_1' - B_2'\right) B_2' = - M \left[1 - \frac{1}{4\, s_g} + \cdots\right] \qquad \text{(B-9)}$$

Equation (B-9) plus the assumption of constant damping rate near the summit allows us to write

$$\hat{\xi}_G = \hat{\xi}_g + \frac{1}{M} e^{B_2(s) - B_{2a}} \int_0^s G'(\hat{s}) e^{-[\lambda_2(\hat{s} - s_a) + i\phi_2(\hat{s})]} d\hat{s} \qquad \text{(B-10)}$$

Equation (B-10) is equivalent to Equation (3.6).